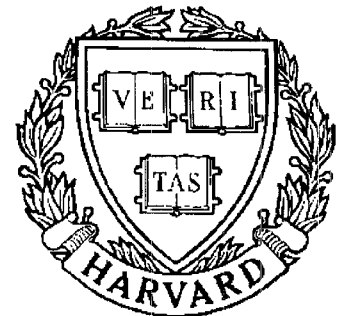


# TECHNICAL RESEARCH REPORT



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## **On the Dynamics of Floating Four-Bar Linkages.**

### **II. Bifurcations of Relative Equilibria**

*by R. Yang and P.S. Krishnaprasad*

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**ON THE DYNAMICS OF FLOATING FOUR-BAR LINKAGES.  
II: BIFURCATIONS OF RELATIVE EQUILIBRIA \***

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**ABSTRACT**

Continuing our program to understand the geometry and dynamics of floating four-bar linkages, we explore the relative equilibria of an assembly that admits symmetric configurations. We show that a symmetric configuration is a relative equilibrium. As we vary certain kinematic parameters which preserve the symmetry, a symmetric relative equilibrium is bifurcated. The type of bifurcations can be either supercritical or subcritical pitchfork. The stability of the relative equilibria at symmetric configurations is investigated. Elementary techniques of singularity theory are applied in the analysis of the bifurcations. This investigation illustrates the possible rich dynamics in multibody systems with closed loop structure even with small number of degrees of freedom.

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## 1. Introduction

An interesting problem in the mechanics of multibody systems is to find relative equilibria. Investigations along this line in several examples, e.g. planar two- or three-body system, two-body system with ball-in-socket joints, etc., have been reported in [2, 6, 9–11]. Increasing sophistication in spacecraft configurations (e.g. the NASA Flight Telerobotic Servicer, modular space station designs, etc.) has underscored the need for a better understanding of the dynamics of multibody systems. Systems incorporating kinematic loops, as does the four-bar linkage, are particularly challenging. The ideas of the present paper might be used for instance in bringing a modular articulated space structure into a nonstandard (e.g. nonsymmetric) equilibrium shape by slowly altering a bifurcation parameter, say, through mass shifting. It is our intention in this paper to illustrate the possibilities for such a *control strategy* by focusing on the bifurcation phenomena in one example, the four-bar linkage. In [12] we studied the geometry and dynamics of floating four-bar linkages which may have a general structure. Relevant notations, terminology, and key concepts are recalled in Appendix A. In this paper, we study *qualitatively* the relative equilibria of a floating four-bar linkage with symmetric configurations.

Referring to the notations in Appendix A, a floating four-bar linkage is of symmetric type if, with proper labels of the bars,

$$m_1 = m_3$$

$$d_{01} = d_{03}, \quad d_{10} = d_{30}, \quad d_{12} = d_{32}, \quad d_{21} = d_{23}. \quad (1.1)$$

In other words it can form symmetric shapes as shown in Fig. 1.1. It is not hard to verify that any four-bar linkage which satisfies (1.1) has two such symmetric configurations.

In this paper we shall show how the relative equilibria of this system vary with respect to certain *parameters*: the positions of body-centers-of-mass. We shall show that the two symmetric configurations are relative equilibria for any choice of the parameters. By varying the offset of the center of mass of the 0-th bar, the symmetric relative equilibria are bifurcated generically. The bifurcations are of pitchfork type. For different choices of position of body-center-of-mass of the 2-nd bar, this pitchfork bifurcation can be either

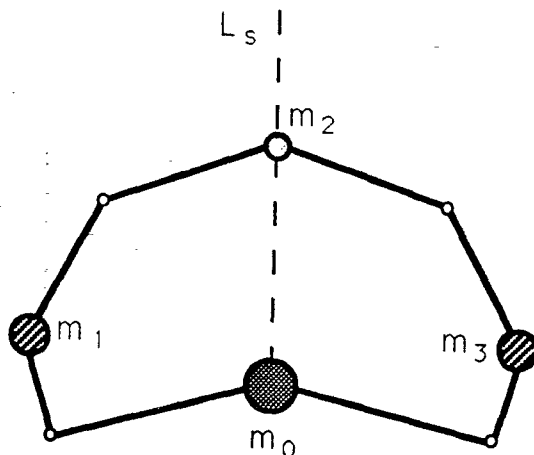


Fig. 1.1 A Symmetric Configuration

*supercritical* or *subcritical*. To avoid tedious calculations, we will consider a simplified case which also exhibits these bifurcation phenomena. In fact, the verification of the assertions in this paper have been carried out both numerically and symbolically. For symbolic analysis, MACSYMA<sup>1</sup> was employed.

As in [12], the theorem of Smale has been used to determine relative equilibria of the system. The singularity theory [3,5] has been applied in the analysis of the bifurcation problem.

## 2. Relative equilibria and bifurcations

We recall Smale's theorem about relative equilibria [8]. Consider a hamiltonian system  $(M, \omega, X_H)$  where  $M$  is a  $2n$ -dimensional smooth manifold,  $\omega$  is a symplectic

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<sup>1</sup> MACSYMA is a trademark of Symbolics Inc., Cambridge, Mass.

form on  $M$ ,  $X_H$  is hamiltonian vector field determined by

$$i_{X_H}\omega = dH$$

for some smooth hamiltonian function  $H$  on  $M$ . Here, we consider natural mechanical systems, so  $M = T^*Q$ , the cotangent bundle of the configuration space  $Q$ , of the system. Let  $G$  be a Lie group acting on  $Q$  on the left,

$$\begin{aligned}\Phi : G \times Q &\rightarrow Q \\ (g, q) &\mapsto \Phi_g(q) \triangleq \Phi(g, q),\end{aligned}$$

and hence on  $T^*Q$  by cotangent lift. The following definition of a relative equilibrium is standard [1].

**Definition 2.1:** Let  $F_{X_H}^t$  be the flow of  $X_H$  on  $M$ . Then  $z_e \in M$  is a relative equilibrium if  $F_{X_H}^t(z_e)$  is a stationary motion, i.e. there exists  $\xi \in \mathcal{G}$  such that

$$F_{X_H}^t(z_e) = \exp(t\xi)(z_e),$$

where  $\mathcal{G}$  is the Lie algebra of the group  $G$ .

**Remark:** A physical interpretation of relative equilibria is that if the dynamics of a system is rotationally invariant, the dynamical orbit of a relative equilibrium appears to be a fixed point for an observer in a suitable uniformly rotating coordinate system.

Given a simple mechanical system with symmetry,  $(Q, K, V, G)$ , where  $K$  is a  $G$ -invariant Riemannian metric on  $Q$ ,  $V : Q \rightarrow R$  is a  $G$ -invariant potential function, we have the following very useful theorem.

**Theorem :** [8] For simple mechanical system with symmetry  $(Q, K, V, G)$ , define

$$V_\xi : Q \rightarrow R : q \mapsto V(q) - \frac{1}{2}K(\xi_Q(q), \xi_Q(q)) \quad (2.1)$$

for each  $\xi \in \mathcal{G}$ , where  $\xi_Q$  is infinitesimal generator of the action corresponding to  $\xi$ . Then  $z_e = (q_e, p_e) \in T^*Q$  is a relative equilibrium if and only if  $q_e$  is a critical point of  $V_\xi$  for some  $\xi \in \mathcal{G}$  and  $p_e = K^b(\xi_Q(q_e))$ .

**Remark:**

- (1). The function  $V_\xi$  is called *augmented potential function*.

(2). It can be shown that, for a given  $\xi \in \mathcal{G}$ ,  $V_\xi$  has the symmetry,

$$V_\xi(\Phi_g(x)) = V_\xi(x)$$

for all  $g \in G_\xi := \{g \in G | Ad_g \xi = \xi\}$ . If  $G$  is abelian,  $G_\xi = G$ . If the action  $\Phi$  is free and proper, then the quotient space,  $Q/G_\xi$ , is a smooth manifold and  $\pi_\xi : Q \rightarrow Q/G_\xi$  is a submersion. Thus  $V_\xi$  induces a function  $\hat{V}_\xi$  on  $Q/G_\xi$  such that

$$V_\xi = \hat{V}_\xi \circ \pi_\xi.$$

(3). If  $G$  is abelian, it can be shown that if  $\pi_\xi(q_e)$  is a local minimizer of  $\hat{V}_\xi$ , the corresponding  $z_e = (q_e, p_e)$  is a stable relative equilibrium, and if  $\pi_\xi(q_e)$  is a local maximizer of  $\hat{V}_\xi$ , the corresponding  $z_e = (q_e, p_e)$  is an unstable relative equilibrium. We will refer to  $Q/G_\xi$  as the *shape space*, and the points  $\pi_\xi(q_e)$  as the relative equilibrium shapes.

(4). One can also use the *amended potential* in the sense of Smale to study the relative equilibria (see [7,8]). ■

Now we return to the problem of relative equilibria of floating four-bar linkages which admit symmetric configurations. In [12] we have shown that a floating four-bar linkage is a simple mechanical system with symmetry with group  $S^1$ . Moreover, setting  $V$  to be zero, locally,  $\hat{V}_\xi$  is of the form

$$\hat{V}_\xi(\theta_{10}) = -\mathbf{e}^T \tilde{\mathbf{J}} \mathbf{e} \tag{2.2}$$

where  $\mathbf{e} = (1 \ 1 \ 1 \ 1)^T$ , the elements of the  $4 \times 4$  matrix  $\tilde{\mathbf{J}}$  are functions of a relative angle, say,  $\theta_{10} \equiv \theta_1 - \theta_0$ . Although it is not easy to find analytically the critical points of (2.2), for a particular example one can easily do this numerically since  $\hat{V}_\xi$  is only a one variable function. Unlike the planar two body problem [9] for which the dimension of the shape space is also one, here the function  $\hat{V}_\xi$  has many parameters, even under the conditions of (1.1). A natural question is to determine how these parameters affect the relative equilibria, e.g. their numbers and location on shape space, etc.. *Of course, it is difficult to answer this question for completely arbitrary choice of parameters.* However, by leaving one particular parameter free such that the assembly preserves its symmetric configurations and



freezing all other parameters, one still can observe a nontrivial bifurcation phenomenon. To illustrate this we consider an example.

**Example:** Let us choose the parameters as follows.

$$m_0 = 1, \quad m_1 = 1, \quad m_2 = 1, \quad m_3 = 1;$$

$$\mathbf{d}_{03} = (-2, \lambda_0), \quad \mathbf{d}_{01} = (-2, \lambda_0), \quad \mathbf{d}_{10} = (-1.5, -1), \quad \mathbf{d}_{12} = (1.5, -1),$$

$$\mathbf{d}_{21} = (-1.5, -1.4), \quad \mathbf{d}_{23} = (1.5, -1.4), \quad \mathbf{d}_{32} = (-1.5, -1), \quad \mathbf{d}_{30} = (1.5, -1).$$

Now the assembly has *non-Grashof* structure. We know that  $\theta_{10}$  cannot be used as a global coordinate system since for each  $\theta_{10}$  there are two points in configuration space, which has been shown to be a torus [4]. One point corresponds to *leading form*. Another one corresponds to *lagging form* (see Appendix A). Nevertheless, one can see that  $\theta_{10}$  taken together with the sign of  $\sin(\theta_{23})$  can separate these forms. For this reason, we define a new variable as follows. Assume the 0-th bar is the longest bar. Let  $\alpha > 0$  be the maximum attainable interior angle between 0-th bar and 1-st bar. Then  $\hat{\theta}$  is defined as follows:

$$\hat{\theta} = \begin{cases} -\frac{\theta_{10}-\pi-\alpha}{\alpha} \frac{\pi}{2}, & \text{if } \sin(\theta_{23}) \geq 0 \text{ (leading form);} \\ \frac{\theta_{10}-\pi-\alpha}{\alpha} \frac{\pi}{2}, & \text{if } \sin(\theta_{23}) < 0 \text{ (lagging form).} \end{cases}$$

It is clear that if we identify  $-\pi$  and  $\pi$  to be the same point on  $S^1$ ,  $\hat{\theta}$  parameterizes  $S^1$ .

Now, using  $\hat{V}_\xi$ , for any  $\lambda_0$  one can find relative equilibria  $(\theta_{10})_e$  for both leading form and lagging form, and hence one can find  $\hat{\theta}$ . As  $\lambda_0$  changes from  $-\infty$  to  $+\infty$ , one can plot a diagram with  $\lambda_0$  and  $\hat{\theta}$  as the parameter. Fig. 2.1 shows the result, in which solid dots represent stable relative equilibria, small circles represent unstable relative equilibria.

From this example one can make the following empirical observations:

(1). There are two unbounded symmetric branches on the diagram and these branches are bifurcated at some points. The bifurcations appear to be pitchfork bifurcations.

(2). Almost any value of  $\hat{\theta}$  can be relative equilibrium for a particular  $\lambda_0$ . In other words the bifurcation diagram is connected globally.

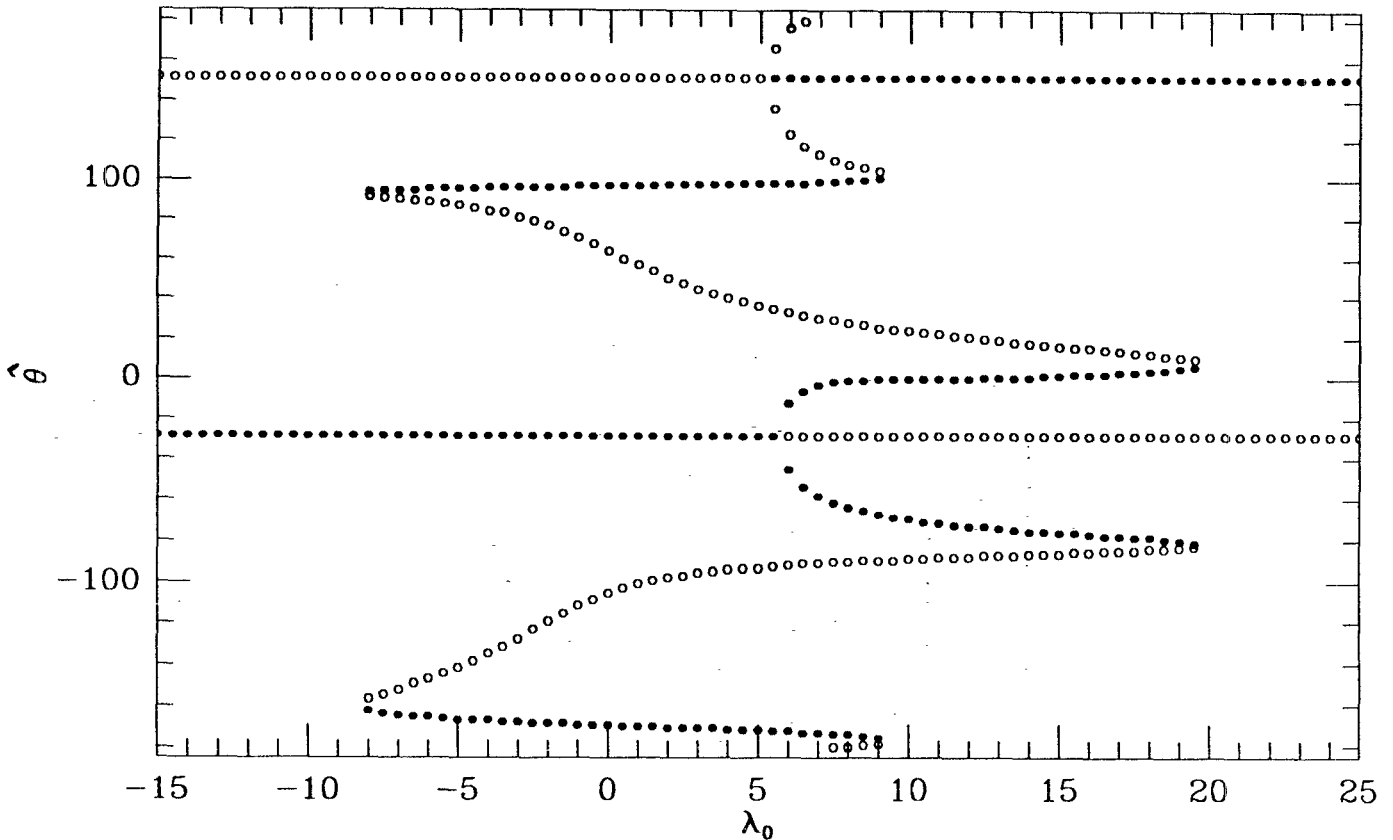


Fig. 2.1 Bifurcation diagram: an example

(3). The number of relative equilibria can be two, six and ten.

The first observation, which relates to the local bifurcation problem, is what we will concentrate on in the rest of this paper. The others will be discussed in section 4.

### 3. The recognition problem in bifurcation theory

Our problem raised in the previous section can be reduced to showing how the solutions  $x$  of a single scalar equation

$$g(x, \lambda) = 0 \tag{3.1}$$

change with the parameter  $\lambda$ ; or, more precisely, what type of bifurcation occurs with parameter  $\lambda$ . Without loss of generality, one can assume  $g(0,0) = 0$ . Moreover, we assume  $g : R \times R \rightarrow R$  is smooth. This is one of the standard local (static) bifurcation problems with one state variable, called *the recognition problem*. As with many bifurcation problems, this problem can be solved successfully through singularity theory, in which the

related issue is called *finite determinacy*. Let  $z = (x, \lambda)$ . Near origin, the function  $g$  can be written as

$$g(z) = \sum_{|\alpha| < k+1} \frac{1}{\alpha!} \left( \frac{\partial}{\partial z} \right)^\alpha g(0) z^\alpha + \sum_{|\alpha| = k+1} a_\alpha(z) z^\alpha, \quad (3.2)$$

for some smooth functions  $a_\alpha$  defined in a neighborhood of the origin. Here we used the conventions with multi-indices:

$$|\alpha| = \alpha_1 + \alpha_2, \quad \alpha! = (\alpha_1)! (\alpha_2)!,$$

$$z^\alpha = x^{\alpha_1} \lambda^{\alpha_2}, \quad \left( \frac{\partial}{\partial z} \right)^\alpha = \left( \frac{\partial}{\partial x} \right)^{\alpha_1} \left( \frac{\partial}{\partial \lambda} \right)^{\alpha_2}.$$

A key question is what terms in Eq.(3.2) can be ignored such that the values of coefficients of remaining terms can be used to determine the qualitative behavior of the original equation (3.1), for example, the variation in number of solutions. Singularity theory solves this problem by finding a suitable change of coordinates such that function  $g$  is equivalent to a standard model  $h$ , called *normal form*. A precise definition is given as follows (see [5]).

**Definition 3.1:** Two smooth mappings  $g, h : R \times R \rightarrow R$  defined near the origin are *equivalent* if there exist a local diffeomorphism of  $R^2$ ,  $(x, \lambda) \mapsto (X(x, \lambda), \Lambda(\lambda))$  at the origin and a nonzero function  $S(x, \lambda)$ , such that

$$g(x, \lambda) = S(x, \lambda) h(X(x, \lambda), \Lambda(\lambda)) \quad (3.3)$$

where  $X_x(0, 0) > 0$  and  $\Lambda'(0) > 0$ . If  $\Lambda = \lambda$ ,  $g$  and  $h$  are *strongly equivalent*.

From this definition we see that, since  $S(x, \lambda)$  is nonzero, the solution of  $g(x, \lambda) = 0$  and  $h(X, \Lambda) = 0$  are the same in the sense of diffeomorphism. From this point of view, by means of singularity theory one can show why and what the high-order terms in Eq.(3.2) do not effect the qualitative behavior of equation  $g(x, \lambda) = 0$ . It should be noticed that although this method does not tell us how to derive an appropriate normal form  $h$ , for most physical problems, such as the one considered in this paper, it is not hard to pick up some of the candidates from a large number of known simple polynomials of  $x$  and  $\lambda$ , or the model of normal forms which have standard bifurcation diagrams. This is in essence the spirit of application of singularity theory to a physical problem.

Without considering detailed issues of singularity theory which are applicable to bifurcation problems, we directly give the following result which will be used in the next section. For details see [5], chapter 2. First we need the concept of germs.

**Definition 3.2:** Two smooth functions defined near the origin are *equivalent as germs* if there is some neighborhood of the origin on which they coincide. Let  $\mathcal{E}_{x,\lambda}$  denote the set of equivalence classes of such functions. The elements in  $\mathcal{E}_{x,\lambda}$  are called *germs*.

**Lemma:** A germ  $g \in \mathcal{E}_{x,\lambda}$  is strongly equivalent to

$$\epsilon x^k + \delta \lambda x \quad (3.4)$$

for  $k > 2$  if and only if at  $x = \lambda = 0$

$$g = \frac{\partial}{\partial x} g = \cdots = \left(\frac{\partial}{\partial x}\right)^{k-1} g = \frac{\partial}{\partial \lambda} g = 0 \quad (3.5)a$$

and

$$\epsilon = \text{sgn}\left(\frac{\partial}{\partial x}\right)^k g, \quad \delta = \text{sgn} \frac{\partial}{\partial x} \frac{\partial}{\partial \lambda} g. \quad (3.5)b$$

**Remark:** When  $k = 3$  the normal form (3.4) provides a pitchfork bifurcation. From this lemma, so is  $g$  if (3.5) holds. It is easy to show that if  $\epsilon\delta > 0$ , the pitchfork bifurcation is subcritical; if  $\epsilon\delta < 0$ , the pitchfork bifurcation is supercritical.

#### 4. Bifurcations at symmetric configuration

As we have seen, the function  $\hat{V}_\xi$  of a four-bar linkage is a multiple parameter function. One might expect very complicated bifurcation features with respect to these parameters. Here, instead of considering a general structure, we study a special assembly which has symmetric configuration as defined in section 2. To avoid too many tedious calculations we particularly choose the parameters of the assembly as follows.

$$m_0 = m_1 = m_2 = m_3 = 1;$$

$$\mathbf{d}_{03} = (-d_0, \lambda_0), \quad \mathbf{d}_{01} = (d_0, \lambda_0), \quad \mathbf{d}_{10} = (-1, 0), \quad \mathbf{d}_{12} = (1, 0) \quad (4.1)$$

$$\mathbf{d}_{21} = (-d_2, \lambda_2), \quad \mathbf{d}_{23} = (d_2, \lambda_2), \quad \mathbf{d}_{32} = (-1, 0), \quad \mathbf{d}_{30} = (1, 0).$$

where  $d_0$  and  $d_2$  are fixed and  $d_0 > d_2 > 0$ ,  $\lambda_0, \lambda_2 \in R$ . Moreover, we consider the non-Grashof case only, i.e.

$$d_0 + d_2 + 2 - 2(\max\{d_0, d_2, 1\} + \min\{d_0, d_2, 1\}) < 0. \quad (4.2)$$

Figure 4.1 shows two symmetric configurations for above choice of parameters. We will see that although only two parameters  $\lambda_0$  and  $\lambda_2$  are left to be free, the bifurcation features with respect to these parameters are still informative.

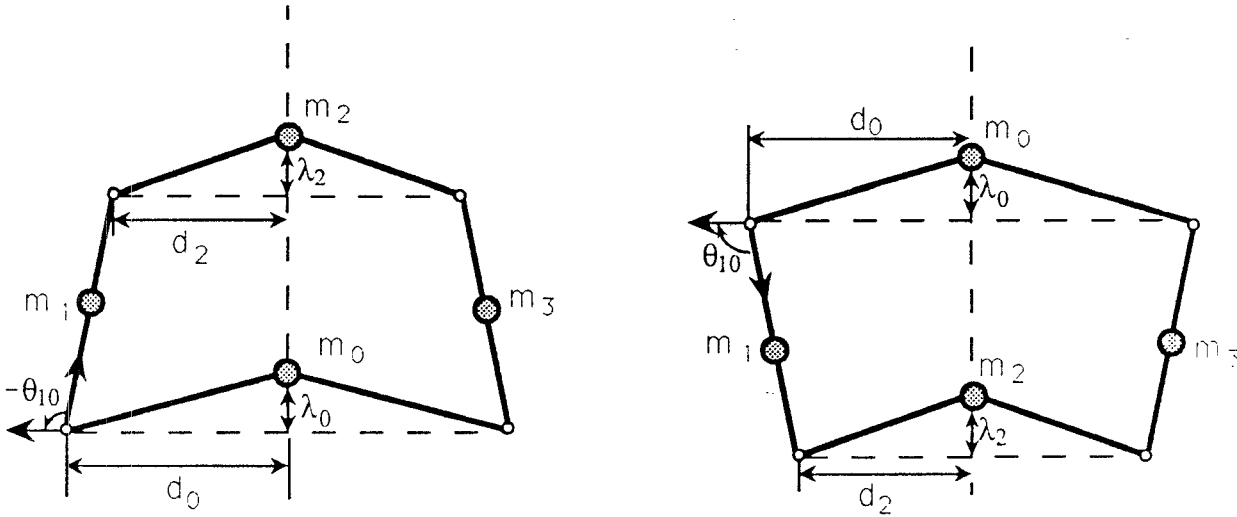


Fig. 4.1 Symmetric Configurations

The function  $\hat{V}_\xi$  now has the following form:

$$\begin{aligned} \hat{V}_\xi = & \frac{1}{4}(2\lambda_2 \sin(\theta_{32}) + d_2 \cos(\theta_{32}) - \cos(\theta_{31}) + d_0 \cos(\theta_{30}) \\ & + 2\lambda_2 \sin(\theta_{21}) + d_2 \cos(\theta_{21}) - d_0 d_2 \cos(\theta_{20}) + d_0 \cos(\theta_{10})) \end{aligned}$$

$$+\frac{\lambda_0}{4}(2\sin(\theta_{30}) + \lambda_2\cos(\theta_{20}) - 2\sin(\theta_{10})) + C \quad (4.3)$$

where  $C$  is a constant determined by  $d_{ij}$ ,  $m_i$  and the moments of inertia of the bodies,  $I_i$ . For  $i, j = 0, 1, 2, 3$ ,  $\theta_{ij} = \theta_i - \theta_j$  and  $\theta_i$  satisfy the constraint equations

$$d_0 + \cos(\theta_{10}) + d_2\cos(\theta_{20}) + \cos(\theta_{30}) = 0 \quad (4.4)a$$

$$\sin(\theta_{10}) + d_2\sin(\theta_{20}) + \sin(\theta_{30}) = 0. \quad (4.4)b$$

In the following, at symmetric configuration, the variables will be denoted by superscript “s” (say,  $\theta_{10}^s$ ), the formulas will be denoted by “|<sub>s</sub>” (say,  $f(\theta_{10})|_s$ ). As shown in the example in section 2, the bifurcation diagram of relative equilibria will be parameterized by  $(\hat{\theta}, \lambda_0)$ .

**Theorem 4.1:** *For a floating four-bar linkage with parameters shown in Eq.(4.1), the bifurcation diagram of relative equilibria has the following properties:*

(1). *There are two infinite branches in the diagram, one corresponds to  $\theta_{10}^s$  in the leading form, another one corresponds to  $\theta_{10}^s$  in the lagging form. We refer to these as the symmetric branches.*

(2). *There exists a constant  $\lambda_2^*$  such that no bifurcation occurs on the symmetric branch of leading form if  $\lambda_2 = \lambda_2^*$ ; and, no bifurcation occurs on the symmetric branch of lagging form if  $\lambda_2 = -\lambda_2^*$ .*

(3). *On the symmetric branch of leading (lagging) form, if  $\lambda_2 < \lambda_2^*$  ( $-\lambda_2^*$ ), there exists a constant  $c_1$  ( $c_3$ ) such that the relative equilibria are stable for  $\lambda_0 < c_1$  ( $c_3$ ), unstable for  $\lambda_0 > c_1$  ( $c_3$ ), bifurcated for  $\lambda_0 = c_1$  ( $c_3$ ); on the other hand, if  $\lambda_2 > \lambda_2^*$  ( $-\lambda_2^*$ ), there exists a constant  $c_2$  ( $c_4$ ) such that the relative equilibria are unstable for  $\lambda_0 < c_2$  ( $c_4$ ), stable for  $\lambda_0 > c_2$  ( $c_4$ ), bifurcated for  $\lambda_0 = c_1$  ( $c_4$ ).*

(4). *Assume  $\lambda_2 \neq \pm\lambda_2^*$ . Let*

$$\epsilon^\pm = \operatorname{sgn} \frac{\pm\epsilon_1\lambda_2^2 + \epsilon_2\lambda_2 \pm \epsilon_3}{\epsilon_4\lambda_2 \pm \epsilon_5} \quad (4.5)$$

and

$$\delta^\pm = \operatorname{sgn}(\delta_1\lambda_2 \pm \delta_2) \quad (4.6)$$

where  $\epsilon_i$  and  $\delta_i$  are constants which are determined by  $d_0$  and  $d_2$ . Then, the bifurcation on the symmetric branch of leading form will be supercritical pitchfork if  $\epsilon^+ \delta^+ < 0$ . It will be subcritical pitchfork if  $\epsilon^+ \delta^+ > 0$ . Similarly, the bifurcation on the symmetric branch of lagging form will be supercritical pitchfork if  $\epsilon^- \delta^- < 0$ . It will be subcritical pitchfork if  $\epsilon^- \delta^- > 0$ .

**Remark:**

(a). Based on the techniques of bifurcation theory shown in section 3, the proof of the above assertions is very elementary. However, two aspects have to be considered before the bifurcation theory is applied. First, one has to determine the *existence* of the bifurcation and where the branches are bifurcated. Second, as one can see, it is not easy to write down explicitly the function  $\hat{V}_\xi$  as a function of one variable. Therefore, when applying the techniques of bifurcation theory which involves as high as fourth order derivatives of the function  $\hat{V}_\xi$ , one should consider the constraint equations simultaneously.

(b). Although the proof of the above assertions is elementary, it requires a large effort in calculations. We used MACSYMA to handle these computations (see listings at the end of the paper). In the following, we only give a sketch of the proof.

**Proof of theorem 4.1:** Note that the function  $\hat{V}_\xi$  can be written as a function of relative angles  $\theta_{10}$ ,  $\theta_{20}$ , and  $\theta_{30}$ , which are related through constraint equations (4.4). From (4.4), we can consider (locally)  $\theta_{20}$  and  $\theta_{30}$  as the functions of  $\theta_{10}$ . Again, from (4.4) one can generate the quantities  $\frac{\partial^i \theta_{20}}{\partial \theta_{10}^i} |_s$  and  $\frac{\partial^i \theta_{30}}{\partial \theta_{10}^i} |_s$  for any positive integer  $i$ . Moreover, from Fig. 4.1 it is easy to see that at symmetric configuration

$$\theta_{20}^s = \pi \quad \text{and} \quad \theta_{30}^s = -\theta_{10}^s. \quad (4.7)$$

With above considerations, one can have closed form expressions for  $\frac{\partial^i \theta_{20}}{\partial \theta_{10}^i} |_s$  and  $\frac{\partial^i \theta_{30}}{\partial \theta_{10}^i} |_s$ .

For instance,

$$\begin{cases} \frac{\partial \theta_{20}}{\partial \theta_{10}} |_s = \frac{2}{d_2} \cos(\theta_{10}^s), \\ \frac{\partial \theta_{30}}{\partial \theta_{10}} |_s = 1 \end{cases} \quad (4.8)$$

and

$$\begin{cases} \frac{\partial^2 \theta_{20}}{\partial \theta_{10}^2} |_s = \frac{2 \cos^2(\theta_{10}^s)}{d_2^2 \sin(\theta_{10}^s)} (d_2 - 2 \cos(\theta_{10}^s)), \\ \frac{\partial^2 \theta_{30}}{\partial \theta_{10}^2} |_s = \frac{2 \cos(\theta_{10}^s)}{d_2 \sin(\theta_{10}^s)} (d_2 - 2 \cos(\theta_{10}^s)) \end{cases} \quad (4.9)$$

and so on.

To prove assertion (1) in the statement of the theorem, one needs to show that at symmetric configuration, the equation

$$\frac{\partial \hat{V}_\xi}{\partial \theta_{10}}|_s = 0$$

does not depend on  $\lambda_0$ . Concentrating on the term involving  $\lambda_0$  in  $\hat{V}_\xi$  and applying Eq.(4.7) and (4.8), one can show that the first derivative of that term with respect to  $\theta_{10}$  at symmetric configuration is zero. Since the rest of the terms of  $\frac{\partial \hat{V}_\xi}{\partial \theta_{10}}|_s = 0$  are still functions of  $\theta_{10}^s$ , one can see two infinite symmetric branches in the bifurcation diagram for two different  $\theta_{10}^s$ . Assertion (1) is thus proved.

Applying (4.8) and (4.9), one can show that the second derivative of the function  $\hat{V}_\xi$  at symmetric configuration has the form

$$\frac{\partial^2 \hat{V}_\xi}{\partial \theta_{10}^2}|_s = \frac{\partial^2 \Pi}{\partial \theta_{10}^2}|_s - \lambda_0 \left( \frac{d_2 - 2\cos^3(\theta_{10}^s)}{d_2 \sin(\theta_{10}^s)} + \lambda_2 \frac{\cos^2(\theta_{10}^s)}{d_2^2} \right), \quad (4.10)$$

where  $\Pi$  is the summation of the terms not involving  $\lambda_0$  in  $\hat{V}_\xi$ . It is obvious that when

$$\lambda_2 = \lambda_2^* \triangleq \frac{2d_2 \cos^2(\theta_{10}^s) - d_2^2}{\cos^2(\theta_{10}^s) \sin(\theta_{10}^s)}, \quad (4.11)$$

$\frac{\partial^2 \hat{V}_\xi}{\partial \theta_{10}^2}|_s$  will not depend on  $\lambda_0$ . One can also show that with Eq.(4.11),  $\frac{\partial^2 \hat{V}_\xi}{\partial \theta_{10}^2}|_s \neq 0$  under assumptions (4.1) and (4.2). This means that bifurcation may not occur on either symmetric branch of leading form or symmetric branch of lagging form. Note that on these different forms  $\cos(\theta_{10}^s)$  has the same value,  $\sin(\theta_{10}^s)$  has the same absolute value but different sign. (See Fig. 4.1) Thus, assertion (2) is proved.

As we have known earlier, the stability of relative equilibria depends on the sign of  $\frac{\partial^2 \hat{V}_\xi}{\partial \theta_{10}^2}|_s$ . From Eq.(4.10) we see that  $\frac{\partial^2 \hat{V}_\xi}{\partial \theta_{10}^2}|_s$  is a linear function of  $\lambda_0$ . Using the  $\lambda_2^*$  in (4.11), the proof of (3) is straight forward.

To prove assertion (4), we apply the bifurcation theory mentioned in section (3). Let  $\lambda_0^*$  denote  $c_i$  in assertion (3) for some suitable  $i$ . One can show that at  $(\theta_{10}, \lambda_0^*)$ ,

$$\frac{\partial \hat{V}_\xi}{\partial \theta_{10}} = \frac{\partial^2 \hat{V}_\xi}{\partial \theta_{10}^2} = \frac{\partial^3 \hat{V}_\xi}{\partial \theta_{10}^3} = \frac{\partial^2 \hat{V}_\xi}{\partial \theta_{10} \partial \lambda} = 0 \quad (4.12)$$



Moreover

$$\frac{\partial^4 \hat{V}_\xi}{\partial \theta_{10}^4}(\theta_{10}^s, \lambda_0^*) = \frac{\pm \epsilon_1 \lambda_2^2 + \epsilon_2 \lambda_2 \pm \epsilon_3}{\epsilon_4 \lambda_2 \pm \epsilon_5} \quad (4.13)$$

and

$$\frac{\partial^3 \hat{V}_\xi}{\partial \theta_{10}^2 \partial \lambda_0}(\theta_{10}^s, \lambda_0^*) = \delta_1 \lambda_2 \pm \delta_2 \quad (4.14)$$

for some constants  $\epsilon_i$  and  $\delta_i$ , where “+” corresponds to leading form, “−” corresponds to lagging form. See Appendix B for expressions of  $\epsilon_i$  and  $\delta_i$ . Since (4.13) and (4.14) are not zero in general, applying the lemma in section 3, we can say  $\frac{\partial \hat{V}_\xi}{\partial \theta_{10}}$  is strongly equivalent to the normal form of pitchfork bifurcation. In addition, the type of pitchfork bifurcation depends on the sign of (4.13) and (4.14). The assertion (4) is proved. ■

**Remark:** (i). The condition of  $\lambda_2 \neq \lambda_2^*$  guarantees that (4.14) and the denominator of (4.13) are not zero. (ii). In general the bifurcation changes from a supercritical one to subcritical one at the roots of numerator of (4.13).

**Example:** To see how  $\lambda_2$  changes the bifurcation diagram with parameter  $\lambda_0$  we give following example. We will concentrate on the symmetric branch with respect to leading form. Let  $d_0 = 2$  and  $d_2 = 1$ . Then  $\epsilon$  and  $\delta$  have the following form

$$\epsilon^+ = \text{sgn} \frac{\lambda_2^2 + 6.938\lambda_2 - 3.623}{1.873 - 0.051\lambda_2}$$

$$\delta^+ = \text{sgn}(1.025 - 0.028\lambda_2)$$

Then

$$\epsilon^+ \delta^+ = \begin{cases} -1, & \text{if } -7.426 < \lambda_2 < 0.488; \\ +1, & \text{otherwise.} \end{cases}$$

Note that the region for  $\lambda_2$  is an approximation. So we can say that, when  $\lambda_2 \in (-7.426, 0.488)$ , the pitchfork bifurcation is supercritical. Otherwise, it is subcritical. Fig 4.2 shows this result.

Before closing this section, we would like to make some further remarks regarding this paper.

(i). Although our discussion only concentrated on a structure of non-Grashof type, a version of theorem 4.1 also holds for the Grashof case.

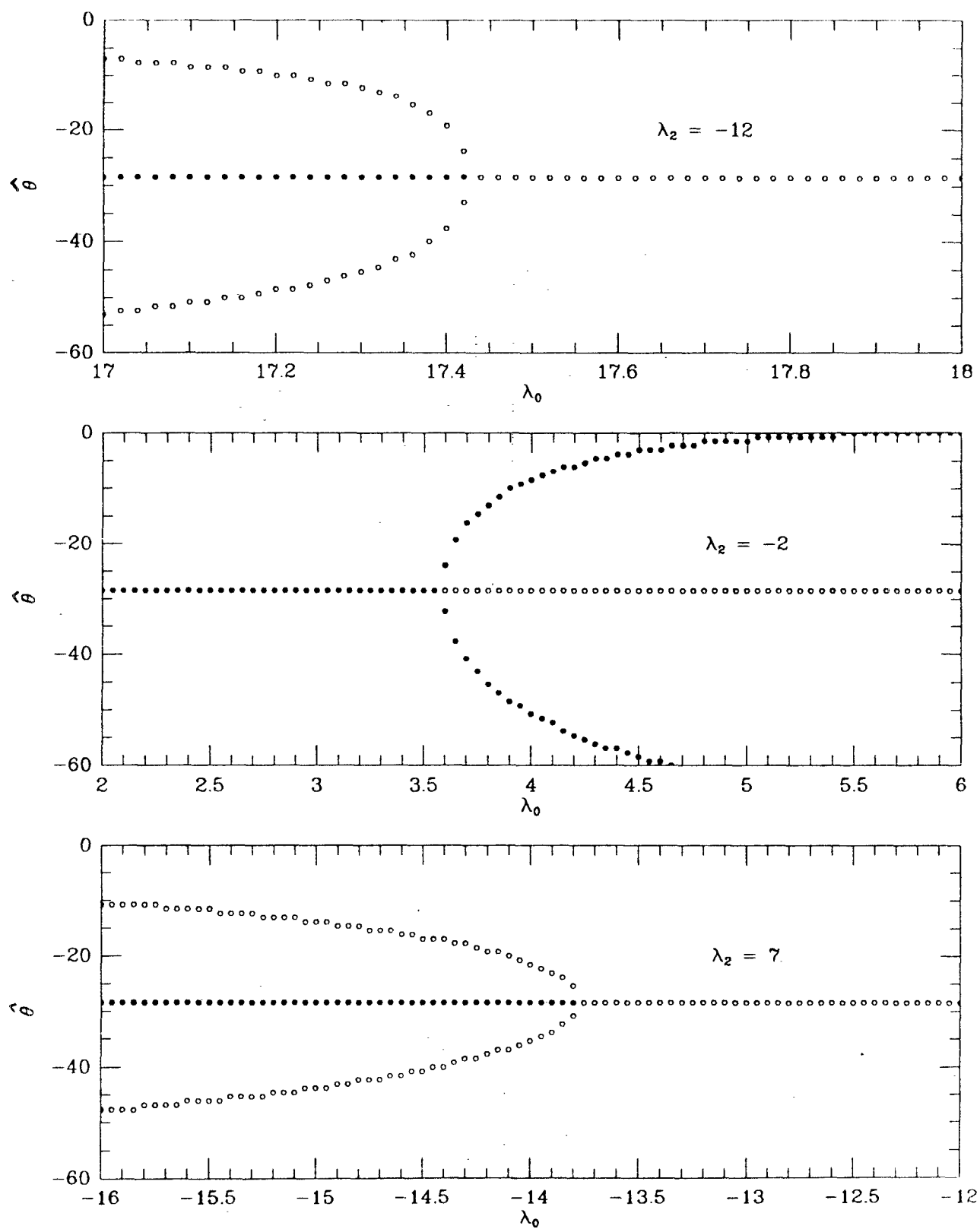


Fig. 4.2 Bifurcation Diagrams

(ii). Up to now we have understood the phenomenon of bifurcation on the symmetric branches. The global analysis of the bifurcations is in progress. This involves massive symbolic computations. However, as shown in the example in section 2 and other simulations, one can numerically determine a global bifurcation diagram. A large body of such numerical simulations show that the branches in the bifurcation diagram are connected. In other words, for any point in shape space, there is a finite  $\lambda_0$  for which that shape determines a relative equilibrium. This is consistent with the linearity of  $\hat{V}_\xi$  in  $\lambda_0$ . This property provides a possibility to control the attitude of a space structure with a closed kinematic chain by simply changing the position of the center of mass of some bars.

(iii). Our results in this paper rely on some ideal conditions, for instance, the symmetry conditions (1.1), and the absence of external and internal disturbances. One may ask what will happen when these conditions are violated. The answer to this question will be related to the notion of *universal unfolding* in bifurcation theory. We will explore this in a further paper. Numerical results show that it is possible to use the unfolding property to control the shape of the structure near the bifurcation point.

## 5. Conclusion

We have investigated the local and global structure of bifurcations of relative equilibria in a model multibody problem – the floating four-bar linkage. The observed global connectivity of the bifurcation diagram suggests that it might be possible to change the attitude/orientation/shape of such a system by merely tuning an appropriate parameter, e.g. a mass offset. The precise dynamics associated to such a control strategy would, of course, depend on the role of internal dissipation and consequent change of a stable center into a focus. Further work on the phase portrait and domains of attraction is under way and we hope to report on these matters in a future paper.

Since we employed the symbol manipulation language MACSYMA for our local analysis, we have included a listing of the relevant MACSYMA program at the end of this paper, to assist the reviewer.

## 6. Acknowledgement

A preliminary version of the results of this paper was presented by P. S. Krish-

naprasad at the Theoretical and Applied Mechanics Colloquium at Cornell University. He gratefully acknowledges the comments of Tim Healey and Jerrold Marsden and also the warm hospitality of the Mathematical Sciences Institute and the Department of Theoretical and Applied Mechanics during the Fall semester 1989.

## Appendix A

### A.1. Notations and Geometric Constraints

The structure of a closed floating four-bar linkage is represented in Fig. A.1. The bars are labeled clockwise from 0 to 3 as shown. We define the following quantities.

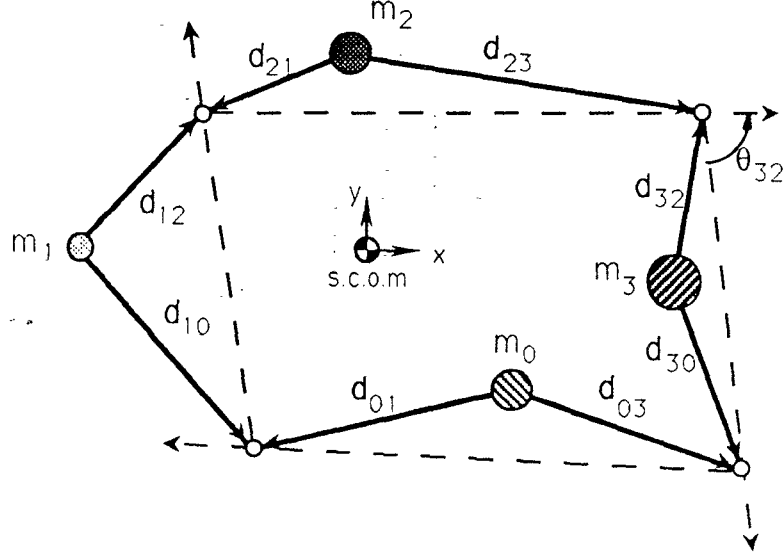


Fig. A.1 The general structure of four-bar linkage

$d_{ij}$  the vector of hinge point which connects  $i$ -th bar with  $j$ -th bar relative to a body-fixed frame with origin at the center of mass of the  $i$ -th body;

$R(\theta_i)$  the rotation through angle  $\theta_i$  giving the orientation of  $i$ -th bar relative to the inertial space;

$$R(\theta_i) = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{pmatrix};$$

$R(\theta_{ij})$  joint rotation between  $i$ -th and  $j$ -th bar,

$$R(\theta_{ij}) = R(\theta_i - \theta_j) = R(\theta_i)R(-\theta_j);$$

$l_i$  the distance between the joints on  $i$ -th bar, (or "length" of  $i$ -th bar); all  $l_i > 0$ ;  $l_i = \|\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}\|$ ;

With above notations, the loop constraint can be represented as

$$\sum_{i=0}^3 R(\theta_i)(\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}) = 0,$$

where we adopt the convention that  $\mathbf{d}_{i,4} = \mathbf{d}_{i,0}$  and  $\mathbf{d}_{4,i} = \mathbf{d}_{0,i}$ .

## A.2. The Configuration Space

Let

$s$  = length of shortest bar

$l$  = length of longest bar

$p, q$  = lengths of intermediate bars.

Assume that the observer is at the center of mass of the system. Then we have following results. (See [12])

If

$$s + l < p + q$$

the configuration space is isomorphic to two separated tori. Under this condition the assembly is called *Grashof* type. If

$$s + l > p + q$$

the configuration space is isomorphic to one torus. Under this condition the assembly is called *non-Grashof* type. If

$$s + l = p + q$$

the configuration space is not a smooth manifold.

Let  $l_1 = s$  and define the orientation of bars as shown in Fig A.1. Then we say that the structure is in *leading form* if  $\sin(\theta_3 - \theta_2) < 0$ , in *lagging form* if  $\sin(\theta_3 - \theta_2) > 0$ . From this definition, we see that for the Grashof case, each component of the configuration space corresponds to a distinct form. Two such forms cannot be deformed into each other. But for the non-Grashof case, they can be continuously deformed on the torus.

## Appendix B

The expression of  $\epsilon_i$  and  $\delta_i$  in the theorem 4.1 are of the following form.

$$\epsilon_1 = -12d_0^2(d_2 - d_0)^2(d_2^3 - d_0d_2^2 - d_0^2d_2 - 2d_2 + d_0^3 - 4d_0);$$

$$\epsilon_2 = 48d_0^2d_2((d_2 - d_0)^2 - 1)(d_2^2 - d_0^2)\sqrt{4 - (d_0 - d_2)^2};$$

$$\epsilon_3 = 48d_0^2d_2^2(d_2 - d_0)^2(d_2^3 - d_0d_2^2 - d_0^2d_2 - 4d_2 + d_0^3 - 2d_0);$$

$$\epsilon_4 = d_2^3((d_2 - d_0)^2 - 4)(d_2 - d_0)^2\sqrt{4 - (d_0 - d_2)^2};$$

$$\epsilon_5 = 2d_2^4((d_2 - d_0)^2 - 4)((d_2 - d_0)^3 - 4d_2).$$

Let

$$\Delta = 4d_2^2\sqrt{4 - (d_0 - d_2)^2}.$$

Then

$$\delta_1 = -\frac{1}{\Delta}(d_2 - d_0)^2\sqrt{4 - (d_0 - d_2)^2};$$

$$\delta_2 = -\frac{2}{\Delta}d_2(d_2^3 + 3d_0d_2^2 - 3d_0^2d_2 + 4d_2 + d_0^3).$$

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**The MACSYMA Program  
to Determine Local Bifurcations**

```

/*****
/*  BIFURCATIONS OF RELATIVE EQUILIBRIA  */
*****/

apply(writefile, [bifur_out]);

depends([th_20, th_30], th_10);

/*****
/*  CONSTRAINT EQUATIONS  */
*****/
f1: d_0+cos(th_10)+d_2*cos(th_20)+cos(th_30)$
f2:      sin(th_10)+d_2*sin(th_20)+sin(th_30)$

/*****
/*  THE FIRST DERIVATIVE OF CONSTRAINT  */
/*  EQUATIONS W.R.T. th_10  */
*****/
dlf1:diff(f1, th_10, 1)$
dlf2:diff(f2, th_10, 1)$

/*****
/*  THE EVALUATION OF  */
/*  THE FIRST DERIVATIVE OF CONSTRAINT  */
/*  EQUATIONS W.R.T. th_10 AT  */
/*  SYMMETRIC CONFIGURATIONS  */
*****/
dlf1s:ev(dlf1, diff(th_20, th_10, 1)=dls_2,
          diff(th_30, th_10, 1)=dls_3,
          th_20=%pi, th_30=-ths_10, th_10=ths_10)$
dlf2s:ev(dlf2, diff(th_20, th_10, 1)=dls_2,
          diff(th_30, th_10, 1)=dls_3,
          th_20=%pi, th_30=-ths_10, th_10=ths_10)$

/*****
/*  SOLVE Dth_20/Dth_10 AND Dth_30/Dth_10  */
*****/
solve([dlf1s, dlf2s], [dls_2, dls_3]),globalsolve:true;

/*****
/*  THE SECOND DERIVATIVE OF CONSTRAINT  */
/*  EQUATIONS W.R.T. th_10  */
*****/
d2f1:diff(dlf1, th_10, 1)$
d2f2:diff(dlf2, th_10, 1)$

/*****
/*  THE EVALUATION OF  */
/*  THE SECOND DERIVATIVE OF CONSTRAINT  */
/*  EQUATIONS W.R.T. th_10 AT  */
/*  SYMMETRIC CONFIGURATIONS  */
*****/
d2f1s:ev(d2f1, diff(th_20, th_10, 1)=dls_2, diff(th_30, th_10, 1)=dls_3,
          diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
          th_20=%pi, th_30=-ths_10, th_10=ths_10)$
d2f2s:ev(d2f2, diff(th_20, th_10, 1)=dls_2, diff(th_30, th_10, 1)=dls_3,
          diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
          th_20=%pi, th_30=-ths_10, th_10=ths_10)$

/*****
/*  SOLVE DDth_20/DDth_10 AND DDth_30/DDth_10  */
*****/
solve([d2f1s, d2f2s], [d2s_2, d2s_3]),globalsolve:true;
d2s_2:factor(d2s_2);
d2s_3:factor(d2s_3);

```

```

/*****/
/* THE THIRD DERIVATIVE OF CONSTRAINT */
/* EQUATIONS W.R.T. th_10 */
/*****/
d3f1:diff(d2f1, th_10, 1)$
d3f2:diff(d2f2, th_10, 1)$

/*****/
/* THE EVALUATION OF */
/* THE THIRD DERIVATIVE OF CONSTRAINT */
/* EQUATIONS W.R.T. th_10 AT */
/* SYMMETRIC CONFIGURATIONS */
/*****/
d3f1s:ev(d3f1, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
diff(th_20, th_10, 3)=d3s_2, diff(th_30, th_10, 3)=d3s_3,
th_20=%pi, th_30=-ths_10, th_10=ths_10)$
d3f2s:ev(d3f2, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
diff(th_20, th_10, 3)=d3s_2, diff(th_30, th_10, 3)=d3s_3,
th_20=%pi, th_30=-ths_10, th_10=ths_10)$

/*****/
/* SOLVE DDDth_20/DDDth_10 AND DDDth_30/DDDth_10 */
/*****/
solve([d3f1s, d3f2s], [d3s_2, d3s_3]),globalsolve:true;
d3s_2:factor(d3s_2);
d3s_3:factor(d3s_3);

/*****/
/* THE FOURTH DERIVATIVE OF CONSTRAINT */
/* EQUATIONS W.R.T. th_10 */
/*****/
d4f1:diff(d3f1, th_10, 1)$
d4f2:diff(d3f2, th_10, 1)$

/*****/
/* THE EVALUATION OF */
/* THE FOURTH DERIVATIVE OF CONSTRAINT */
/* EQUATIONS W.R.T. th_10 AT */
/* SYMMETRIC CONFIGURATIONS */
/*****/
d4f1s:ev(d4f1, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
diff(th_20, th_10, 3)=d3s_2, diff(th_30, th_10, 3)=d3s_3,
diff(th_20, th_10, 4)=d4s_2, diff(th_30, th_10, 4)=d4s_3,
th_20=%pi, th_30=-ths_10, th_10=ths_10)$
d4f2s:ev(d4f2, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
diff(th_20, th_10, 3)=d3s_2, diff(th_30, th_10, 3)=d3s_3,
diff(th_20, th_10, 4)=d4s_2, diff(th_30, th_10, 4)=d4s_3,
th_20=%pi, th_30=-ths_10, th_10=ths_10)$

/*****/
/* SOLVE DDDDth_20/DDDDth_10 AND DDDDth_30/DDDDth_10 */
/*****/
solve([d4f1s, d4f2s], [d4s_2, d4s_3]),globalsolve:true;
d4s_2:factor(d4s_2);
d4s_3:factor(d4s_3);

/*****/
/* DERIVE V FUNCTION */
/*****/
d_03:matrix([-d_0], [mu_0])$
d_01:matrix([d_0], [mu_0])$
d_10:matrix([-1], [0])$

```

```
d_12:matrix([1],[0])$
d_21:matrix([-d_2],[mu_2])$
d_23:matrix([d_2],[mu_2])$
d_32:matrix([-l],[0])$
d_30:matrix([1],[0])$
r_10:matrix([cos(th_10), -sin(th_10)],
             [sin(th_10), cos(th_10)])$
j_01:- (3/8)*(transpose(d_01).(r_10.d_10))+
      (1/8)*(transpose(d_03).(r_10.d_12))$
j_01:ratsimp(j_01, sin(th_10), cos(th_10))$

r_20:matrix([cos(th_20), -sin(th_20)],
             [sin(th_20), cos(th_20)])$
j_02:- (1/8)*(transpose(d_01).(r_20.d_23))-
      (1/8)*(transpose(d_03).(r_20.d_21))$
j_02:ratsimp(j_02, sin(th_20), cos(th_20))$

r_03:matrix([cos(th_03), -sin(th_03)],
             [sin(th_03), cos(th_03)])$
j_03:- (3/8)*(transpose(d_30).(r_03.d_03))+
      (1/8)*(transpose(d_32).(r_03.d_01))$
j_03:ratsimp(j_03, sin(th_03), cos(th_03))$

r_21:matrix([cos(th_21), -sin(th_21)],
             [sin(th_21), cos(th_21)])$
j_12:- (3/8)*(transpose(d_12).(r_21.d_21))+
      (1/8)*(transpose(d_10).(r_21.d_23))$
j_12:ratsimp(j_12, sin(th_21), cos(th_21))$

r_31:matrix([cos(th_31), -sin(th_31)],
             [sin(th_31), cos(th_31)])$
j_13:- (1/8)*(transpose(d_12).(r_31.d_30))-
      (1/8)*(transpose(d_10).(r_31.d_32))$
j_13:ratsimp(j_13, sin(th_31), cos(th_31))$

r_32:matrix([cos(th_32), -sin(th_32)],
             [sin(th_32), cos(th_32)])$
j_23:- (3/8)*(transpose(d_23).(r_32.d_32))+
      (1/8)*(transpose(d_21).(r_32.d_30))$
j_23:ratsimp(j_23, sin(th_32), cos(th_32))$

v:ratsimp(j_01+j_02+j_03+j_12+j_13+j_23, mu_0)$
v:ev(v, th_03 = -th_30, th_32 = th_30 - th_20,
     th_31 = th_30 - th_10, th_21 = th_20 - th_10);

/*****
/* THE FIRST DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 */
*****/
dlv:diff(v, th_10, 1)$

/*****
/* THE FIRST DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 AT */
/* SYMMETRIC CONFIGURATION */
*****/
dlvs:ev(dlv, diff(th_20, th_10, 1)=dls_2, diff(th_30, th_10, 1)=dls_3,
        th_20=pi, th_30=-th_10, th_10=th_10)$
dlvs:trigexpand(dlvs);

/*****
/* THE SECOND DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 */
*****/
d2v:diff(dlv, th_10, 1)$
```

```

/*****
/* THE SECOND DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 AT */
/* SYMMETRIC CONFIGURATION */
/*****
d2vs:ev(d2v, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
        diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
        th_20=%pi, th_30=-ths_10, th_10=ths_10)$
d2vs:trigexpand(d2vs);

/*****
/* THE THIRD DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 */
/*****
d3v:diff(d2v, th_10, 1)$

/*****
/* THE THIRD DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 AT */
/* SYMMETRIC CONFIGURATION */
/*****
d3vs:ev(d3v, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
        diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
        diff(th_20, th_10, 3)=d3s_2, diff(th_30, th_10, 3)=d3s_3,
        th_20=%pi, th_30=-ths_10, th_10=ths_10)$
d3vs:trigexpand(d3vs);

/*****
/* THE FOURTH DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 */
/*****
d4v:diff(d3v, th_10, 1)$

/*****
/* THE FOURTH DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 AT */
/* SYMMETRIC CONFIGURATION */
/*****
d4vs:ev(d4v, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
        diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
        diff(th_20, th_10, 3)=d3s_2, diff(th_30, th_10, 3)=d3s_3,
        diff(th_20, th_10, 4)=d4s_2, diff(th_30, th_10, 4)=d4s_3,
        th_20=%pi, th_30=-ths_10, th_10=ths_10)$
d4vs:trigexpand(d4vs);

/*****
/* THE SECOND DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 AND */
/* THE FIRST DERIVATIVE OF V FUNCTION */
/* W.R.T. mu_0 */
/*****
d3vmu:diff(d2v, mu_0, 1)$

/*****
/* THE SECOND DERIVATIVE OF V FUCTION */
/* W.R.T. th_10 AND */
/* THE FIRST DERIVATIVE OF V FUNCTION */
/* W.R.T. mu_0 */
/* EVALUATED AT SYMMETRIC CONFIGURATION */
/*****
d3vmus:ev(d3vmu, diff(th_20, th_10, 1)=d1s_2, diff(th_30, th_10, 1)=d1s_3,
        diff(th_20, th_10, 2)=d2s_2, diff(th_30, th_10, 2)=d2s_3,
        th_20=%pi, th_30=-ths_10, th_10=ths_10)$
d3vmus:trigexpand(d3vmus);

```

```

/*****/
/* FIND mus_0 AT WHERE THE BIFURCATION HAPPENS */
/*****/
linsolve([d2vs], [mu_0]),globalsolve:true$
mus_0:trigexpand(mu_0);

/*****/
/* AT BIFURCATION POINTS THE FOLLOWING EQUATIONS */
/* SHOULD HOLD */
/* g(th_10, mus_0) = 0 */
/* Dg_th_10(th_10, mus_0) = 0 */
/* DDg_(th_10, th_10)(th_10, mus_0) = 0 */
/* WHERE g=Dv/Dth_10, th_10 IS th_10 AT */
/* SYMMETRIC CONFIGURATION */
/*****/
trigsimp(ev(d1vs, mu_0=mu_0));
trigsimp(ev(d2vs, mu_0=mu_0));
trigsimp(ev(d3vs, mu_0=mu_0));

/*****/
/* AT BIFURCATION POINTS THE FOLLOWING EQUATIONS */
/* SHOULD HOLD */
/* DDDg_(th_10, th_10, th_10)(th_10, mus_0) not= 0 */
/* DDg_(th_10, mu_0)(th_10, mus_0) not= 0 */
/*****/
d4vs:trigexpand(trigsimp(ev(d4vs, mu_0=mu_0)));
d3vmus:trigexpand(trigsimp(ev(d3vmus, mu_0=mu_0)));

/*****/
/* AT LEADING FORM */
/* MAKE d4vs AND d3vmus AS FUCTIONS OF d_0 AND d_2 */
/*****/
mus1_0:ev(mus_0, sin(th_10)=-sqrt(4.-(d_2-d_0)**2)/2,
          cos(th_10)=-(d_0-d_2)/2)$
mus1_0:ratsimp(mus1_0);
d4vslead:ratsimp(ev(d4vs, sin(th_10)=-sqrt(4.-(d_2-d_0)**2)/2,
                  cos(th_10)=-(d_0-d_2)/2,
                  mus_0=mus1_0));
d3vmuslead:ratsimp(ev(d3vmus, sin(th_10)=-sqrt(4.-(d_2-d_0)**2)/2,
                    cos(th_10)=-(d_0-d_2)/2,
                    mus_0=mus1_0));

/*****/
/* AT LAGGING FORM */
/* MAKE d4vs AND d3vmus AS FUCTIONS OF d_0 AND d_2 */
/*****/
mus2_0:ev(mus_0, sin(th_10)=+sqrt(4.-(d_2-d_0)**2)/2,
          cos(th_10)=-(d_0-d_2)/2)$
mus2_0:ratsimp(mus2_0);
d4vsrag:ratsimp(ev(d4vs, sin(th_10)=+sqrt(4.-(d_2-d_0)**2)/2,
                  cos(th_10)=-(d_0-d_2)/2,
                  mus_0=mus2_0));
d3vmusrag:ratsimp(ev(d3vmus, sin(th_10)=+sqrt(4.-(d_2-d_0)**2)/2,
                    cos(th_10)=-(d_0-d_2)/2,
                    mus_0=mus2_0));

/*****/
/* AN EXAMPLE */
/*****/

ev(mus1_0, d_0=4/3, d_2=1);
ev(mus2_0, d_0=4/3, d_2=1);

ev(d4vslead, d_0=4/3, d_2=1);
ev(d3vmuslead, d_0=4/3, d_2=1);

```

```
ev(d4vslag, d_0=4/3, d_2=1);  
ev(d3vmuslag, d_0=4/3, d_2=1);  
  
apply(closefile, [bifur_out]);
```